Scaling properties of complex networks:
Toward Wilsonian renormalization for complex networks

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Abstract—**Nowadays, scaling methods for general large-scale complex networks have been developed. We proposed a new scaling scheme called "two-site scaling". This scheme was applied iteratively to various networks, and we observed how the degree distribution of the network changes by two-site scaling. In particular, networks constructed by the BA algorithm behave differently from the networks observed in the nature. In addition, an iterative scaling scheme can define a new renormalizing method. We investigated the possibility of defining the Wilsonian renormalization group method on general complex networks and its application to the analysis of the dynamics of complex networks.**

I. INTRODUCTION

In recent years, complex networks have been actively investigated. Thanks to recent increases in computational power, we can deal with huge and complex networks that exist in the real world. One subject of particular interest is the cross-disciplinary nature of complex networks. For example, as pointed out by Barabasi et al., the cinema actors' costarring network and the Internet to the metabolic network of cells have power-law or scale-free properties [1], [2]. To explain the scale-free property, the BA algorithm[3] has been proposed. The BA algorithm is a simple growing network model based on the notion of preferential attachment.

Other than frequently discussed quantities such as the powers of the power-law distributions and the cluster coefficient, we can define numerous quantities characterizing a network, but some of these quantities are difficult to calculate because of computational complexity, and we scarcely know which quantities should be used in the study of complex networks for the realistic applications.

On the other hand, recent studies on complex networks have focused on the dynamics of the network[4]. It is desirable to understand the dynamics of the network from an application point of view, because there are many demands related to the control of the dynamics of various networks. However, these studies depend on the individual dynamic system and the algorithms used to generate complex networks. It is necessary to take a systematic approach to the dynamics of complex networks.

In the present paper, we propose a scaling method called two-site scaling. As mentioned in the following section, scaling is essential for the Wilsonian renormalization method. Using the Wilsonian renormalization method, we can extract information about dynamics, such as critical exponents, systematically, which is expected to by universal in nature and would enables us to classify the dynamics on networks and underlying networks from a dynamics point of view.

II. WILSONIAN RENORMALIZATION GROUP METHOD

Wilsonian renormalization methods[5], [6] are powerful and systematic methods in theoretical physics. The Wilsonian renormalization theory is the theory of the flow of the parameters in the parameter space of dynamic systems. Therefore, we can expect the renormalization method to be a systematic approach to the dynamics of the complex networks. In particular, in condensed matter physics, critical exponents derived by the renormalization method are known empirically to have strong "universality"[7], [6], which requires that the systems have same dimension and that the numbers of states have the same critical exponent. The renormalization method usually deploys scaling of lattices.

Here, there is another important procedure, referred to as rescaling, in the Wilsonian renormalization theory. This is a procedure by which to integrate the dynamics in the subgraphs, which is contracted by scaling, and represent their contributions by renormalizing the parameters of the dynamics, for example, parameters of the Hamiltonian. However, in the present paper, we do not discuss rescaling, but rather focus only on how to define the scaling in complex networks.

Thanks to the universality, the critical exponents of the systems are good measures for classifying dynamic systems on the networks. These classes are called universality classes. For

example, critical exponents for Ising models can be derived numerically by using, for example, the Metropolis method.

Renormalization of some network structural quantities has already been applied to complex networks such as Watts-Strogatz's small world network[8] and the geographical embedded network[9]. These scalings can be defined naturally because these networks are embedded in Euclidean space, and scaling can be defined naturally by the scaling of the Euclidean space. However, in these papers only structural quantities are renormalized. Thus, they are not Wilsonian renormalization procedures.

In the general complex network, there is no such naturally obtained scaling method. Therefore, we should define how to scale these networks. In recent years, scaling methods have been proposed for general complex networks[10], [11]. We are prepared to define renormalization group method for general complex networks.

We define the action of the renormalization group by the two-site scaling method defined in Section III. Note that, as mentioned in Section V-B, further study is needed in order to verify that the proposed two-site scaling method will yield an appropriate renormalization scheme. In Section IV, we apply the proposed two-site scaling method iteratively to various networks, including the BA network, the Internet router network, the AS network, the network of actors' costars, and the protein-protein interaction network.

III. TWO-SITE SCALING

The scaling method for general complex networks used herein is the box covering method, which was proposed by Song-Havlin-Makse[10] and investigated in successive studies [11]. The box covering method divides a network to subgraphs of diameter smaller than the given length. Thus, we can regard these methods as coarse grained. In the terminology of graph theory, this procedure corresponds to contraction[1]. Contraction means that the subgraphs are considered to be one node v and the links between the inside of the subgraph and the outside of the subgraph are considered to be a link to the node v. In the scaling, we divide the network into subgraphs called boxes and contract these boxes.

Two-site scaling is a method of dividing a network into randomly selected pairs of adjacent nodes and contracting these pairs. However, if networks have many degree-1 nodes (referred to herein as leaves), randomly selected pairs tend to be pairs that consist of a leaf and the adjacent node, and contraction is reduced to simply removing one leaf. In the Wilsonian renormalization theory, the information to be extracted from the object systems is important. Since we intend to perform two-site scaling homogeneously over the network and to extract information about the dynamics and the network homogeneously over the entire network, this is inappropriate because it makes the two-site scaling method inhomogeneous in the networks. To avoid this, we propose ignoring leaves and apply this procedure to nodes of degree greater than or equal to 2. For degree-1 nodes (leaves), pairs of leaves having adjacent nodes that are identical are contracted.

[1]In previous papers, only the number of subgraphs was counted, so that this graph theoretical interpretation was not necessary
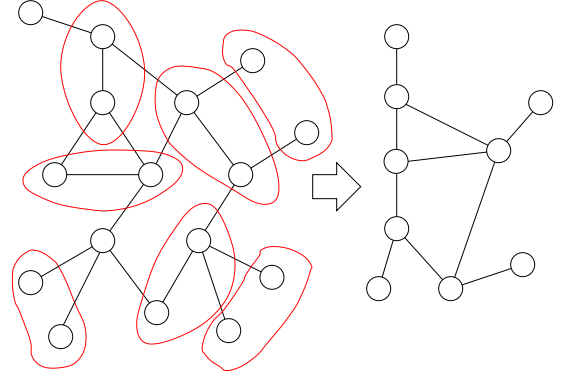

Fig. 1. Example of two-site scaling

Summarizing the above, we have following algorithm.
1) Count adjacent degree-1 nodes for nodes of degree greater than 1
2) Remove degree-1 nodes
3) Fill the network with randomly selected pairs of adjacent nodes
4) Contract the pairs constructed in Step 2.
5) Add half of the number of degree-1 nodes

By this algorithm, the sizes (number of nodes) of the networks without leaves are approximately halved. We can iterate this algorithm and scale networks in a systematic manner. Note that previous scaling methods do not consider this iteration.

A. Comparison with other scaling methods

The method defined above resembles renormalization by block spin transformation (BST) in condensed matter physics. For a d dimensional lattice, the usual method is to divide the lattice into d-dimensional cubes and contract the d-dimensional cube as one site.

However, since in general complex networks, we cannot expect d-dimensional cubes to appear homogeneously, we need another method. The box covering method by [10] divides a network into boxes, where any distance between any pair of nodes in a box is less then ℓ_S. In a simplified version of this method by [10], [11], a node v is selected randomly, and nodes located a distance from v of less than ℓ_B constitute a box.

The box covering method resembles to the scaling method used to calculate the fractal dimension. The intent is not to define the renormalization scheme, but rather to calculate the fractal dimension of complex networks. As a result, there are some problems, such as dispersion of the box sizes, if we employ the box covering method as a scaling method for the renormalization.

The other methods by which to define the renormalization scheme are to embed complex networks into a 1-dimensional lattice or a d-dimensional lattice or to contract a d-dimensional cube[8], [12], [13], [9]. These methods can be extended to the natural extension of renormalization but cannot adapt to general complex networks.
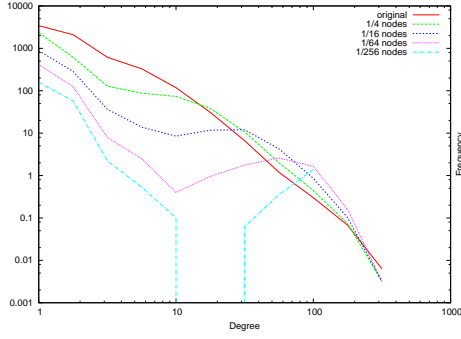
Fig. 2. Scaling of the network by the BA algorithm: scaling property of the degree distribution by every two iterations of two-site scaling
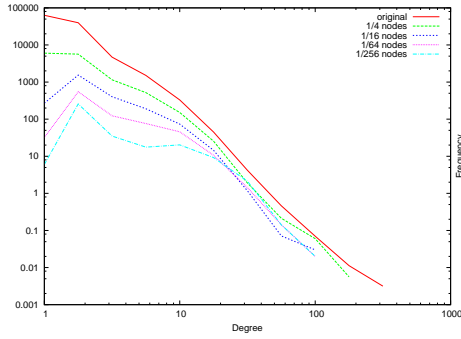

Fig. 4. Scaling of the AS network of the Internet

Fig. 3. Scaling of the router network of the Internet

Fig. 5. Scaling of the actors' costarring network

V. DISCUSSION

The two-site scaling method, which is inspired from block spin transformation in renormalization theory, assigns new quantities to complex networks. We would like to emphasize that the quantities given by scaling methods would influence the dynamics of networks, especially with respect to critical phenomena. Although we could not calculate the critical exponents in the present study, these critical exponents are expected to have a universal property, and would be used to classify the universal classes of dynamic systems on complex networks.

A. Peculiarity of BA networks

We found that the degree distribution of networks constructed by the BA algorithm is transformed from a power-

B. Features of two-site scaling

One characteristic of two-site scaling is that one iteration of two-site scaling is homogeneous contraction, i.e., almost all nodes are included in boxes of the same size and the size of each box is two, which is the minimum number for contraction. This characteristic also minimizes the loss of the fine link structure. Therefore, we can have a coarse-grained network of the required size with less loss of the fine structure of the network.

On the other hand, this may become problematic, as explained above, because special treatment is required for nodes of degree 1. Without this special treatment, we cannot scale the network homogeneously.

IV. SCALING OF DEGREE DISTRIBUTIONS OF VARIOUS NETWORKS

Figure 2, Figure 3, Figure 4, Figure 5, and Figure 6 show the scaling property of the degree distribution of the network constructed by the BA algorithm, the Internet topologies of the router level and the AS level, the actor's costarring network, and protein-protein interaction network, respectively. The two-site scaling method acts iteratively on these networks and the degree distributions of these networks are plotted for every two (one for protein-protein networks) iterations.

As shown in Figure 2, the degree distribution of the network constructed by the BA algorithm shows rapid transition from a power-law distribution to a bimodal distribution. On the other hand, other networks observed in nature maintain a power law distribution.
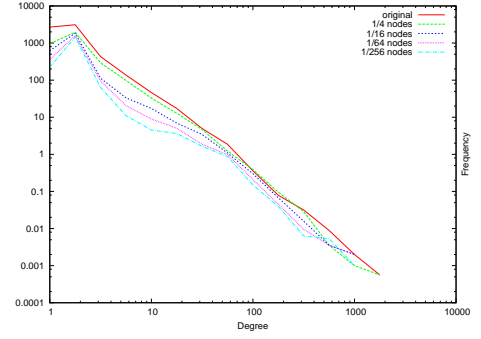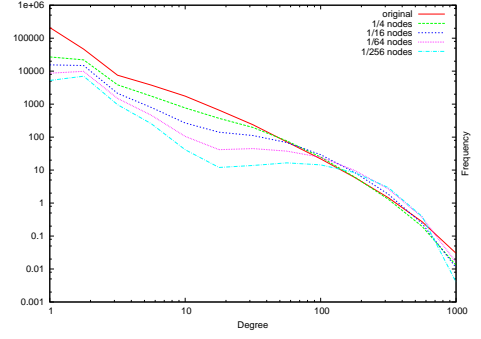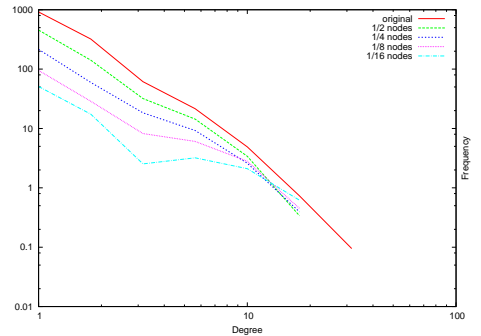

Fig. 6. Scaling of the protein-protein interaction network

law distribution into a bimodal distribution. This is unusual compared to networks observed in nature, such as the Internet, the actor network, and the protein-protein interaction network, which maintain a power law distribution by the iteration of two-site scaling. In other words, networks observed in nature would have more strong scale invariance in terms of not only the degree distribution, but also the proposed scaling method.

The reason for this peculiarity of BA networks remains unknown. This phenomenon should be analyzed in a future study.

B. Toward the Wilsonian renormalization group method

Studies of complex networks are often accompanied by the problem of computational complexity because many elemental problems, such as the isomorphism problem, are known to be NP[14]. Argument by the renormalization method with scaling of the complex network would provide a scalable theory of the dynamics of complex networks.

The two-site scaling method resembles block spin transformation in condensed matter physics. There have been attempts to define the fractal dimension of complex networks[15]. However, the dimension of complex networks cannot be defined in a straightforward manner. Therefore, further investigation is needed in order to verify that the two-site scaling method can define the renormalization group action.

The renormalization scheme defined from the box covering method, which was originally proposed to define the fractal dimension, would be robust with respect to differences in fractal dimension. However, since the iteration of the box covering rapidly contracts a large network to one node, it is difficult to obtain information such as critical exponents from the few iterations of the box covering method. This is because the computation of renormalization should be executed numerically, and the numerical error of critical exponents becomes large.

In future studies, we intend to verify that the renormalization scheme defined by the two-site scaling function is similar to other block spin transformations. This is a difficult problem in that the proposed scaling scheme maps a regular lattice not to a regular lattice, but to complex networks. For this and other reasons, we should employ numerical methods such as Monte Carlo Renormalization. It may be useful to verify that this problem will be overcome by applying this scheme to regular lattices, the universality classes of which are determined. If we have the same critical exponent from two-site scaling method as other renormalization schemes, we can expect these exponents to provide the classification of dynamics and underlying networks.

Moreover, since networks scaled by the BA algorithm have peculiar scaling properties, as compared with natural networks, BA networks have a different structure than natural networks, and this difference may affect the dynamics on BA networks, especially for critical phenomena.

VI. CONCLUSION

We have proposed a scaling method called two-site scaling. This scaling scheme maintains the shape of the degree distribution of various natural networks. However, the degree distribution of BA networks is changed drastically by this scaling method. Networks observed in nature maintain the shapes of their degree distributions by the proposed scaling methods, which means that these networks have stronger scale invariance than scale-free degree distributions. Finally, we discussed the application of the proposed scaling method to the Wilsonian renormalization method. A great deal of research remains to be conducted in order to complete the Wilsonian renormalization theory for complex networks.

Note added: In this paper we proposed a roadmap to the Wilsonian renormalization group method, but we informed that a preprint by F. Radicchi et. el. [16] has been submitted to preprint server just before the presentation on the PHSY-COMNET workshop, but after the acceptance (Dec.1st 2007) of this paper. Although the paper by Radicchi et. el. employs greedy coloring algorithm(GCA) and random burning(RB) for box covering method, insists a similar roadmap to the Wilsonian renormalization group method.

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